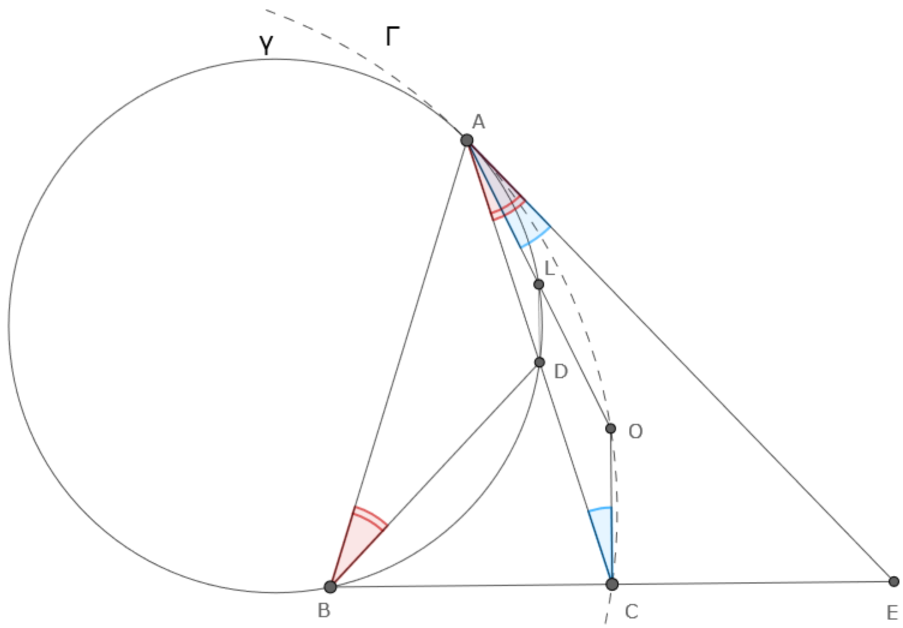
2020 BMO - Problem 1

Problem 1. Let ABC be an acute triangle with $AB = AC$, let D be the midpoint of the side AC, and let γ be the circumcircle of the triangle ABD. The tangent of γ at A crosses the line BC at E. Let O be the circumcentre of the triangle ABE. Prove that the midpoint of the segment AO lies on γ.



Solution 1. We will first prove that C is the midpoint of the segment BE. From the angle equalities

- $\angle BCD = \angle ACB = \angle CBA = \angle EBA$
- $\angle BDC = \angle BAD + \angle DBA = \angle BAD + \angle DAE = \angle BAE$

we can conclude that the triangles $\triangle ABE$ and $\triangle DCB$ are similar.

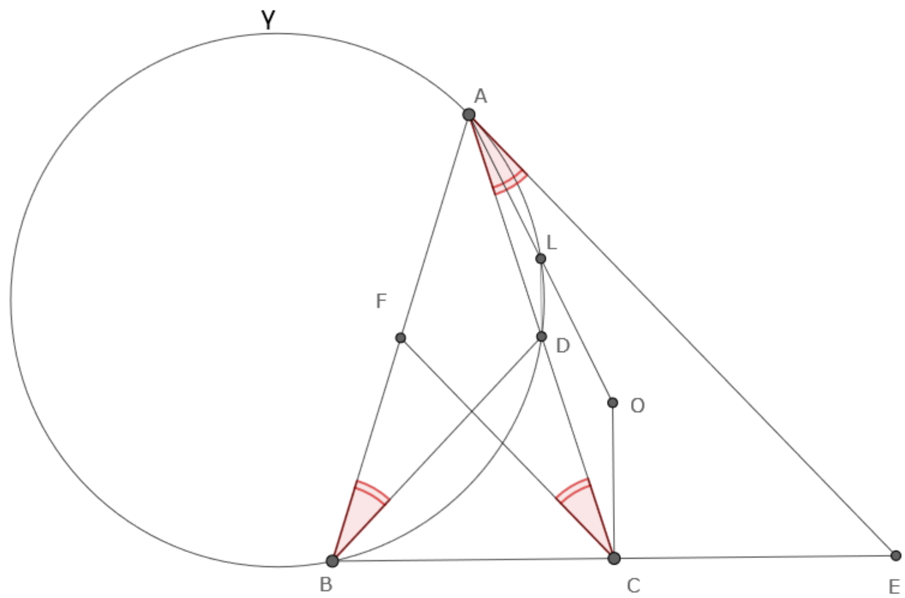
Thus, $BE/BC = AB/CD = 2$, which implies that C is indeed the midpoint of the segment BE.

We will now prove that AE is tangent to the circle ACO. From the angle equalities

- $\angle OAE = 90^\circ - \angle EBA$
- $\angle OCA = \angle OCB - \angle ACB = 90^\circ - \angle CBA = 90^\circ - \angle EBA$

we can conclude that $\angle OAE = \angle OCA$, which implies that AE is indeed tangent to the circle ACO.

Finally, let Γ be the image of γ under the homothethy of center A and factor 2. Clearly, Γ is also tangent to AE at A and passes through C, so Γ must coincide with the circle ACO, which obviously passes through O. Thus, γ passes through the midpoint of the segment AO. $\square$

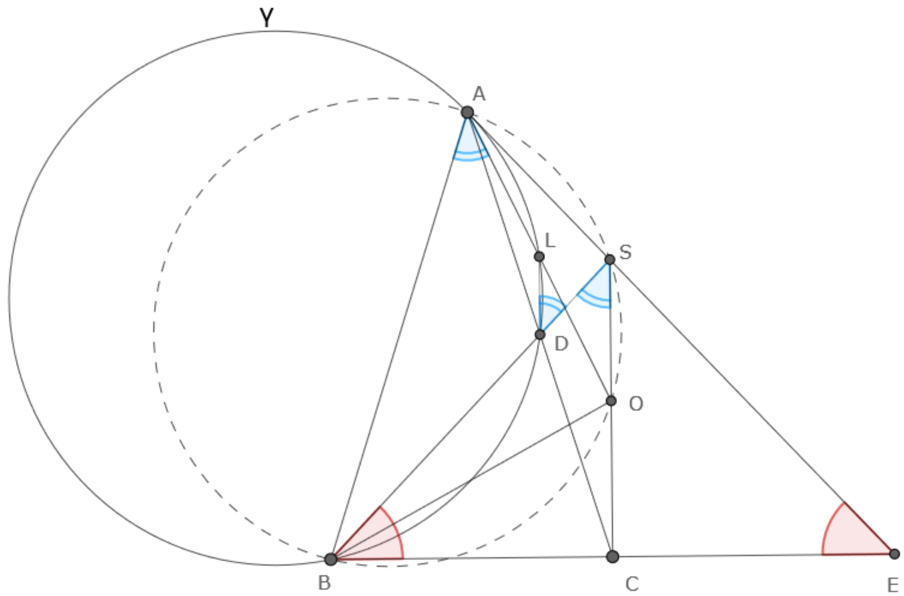


Solution 2. Like in the previous solution, we will first prove that C is the midpoint of the segment BE. Let F be the midpoint of the segment AB. Because $\angle EAC = \angle ABD = \angle FCD$, we have that $CF \| AE$. This implies that CF is a midline in triangle $\triangle BAE$, so C is indeed the midpoint of the segment BE.

Let L be the midpoint of the segment AO. Because LD is a midline in triangle $\triangle AOC$, so $LD \| OC$, which means that $\angle ALD = \angle AOC$. From the angle equalities

- $\angle ALD = \angle AOC = \angle BOC + \angle AOB = \angle BAE + 2\angle BEA$
- $\angle ABD = \angle CEA = \angle BCA - \angle BEA = \angle ABE - \angle BEA$
- $\angle ALD + \angle ABD = \angle BAE + 2\angle BEA + \angle ABE - \angle BEA = 180°$

we obtain that the quadrilateral $ABDL$ is cyclic, thus L lies on γ. □



Solution 3. Like in the previous solutions, establish that C is the midpoint of the segment BE.

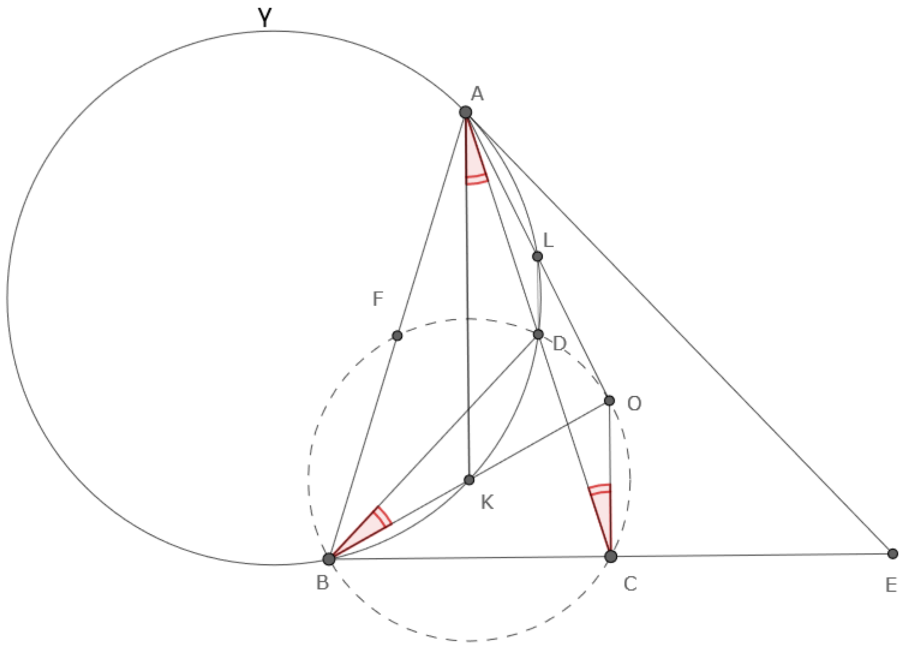
Let S be the intersection between BD and AE. We will first show that S also lies on CO. Because

$$\angle BDC = \angle BAD + \angle DBA = \angle BAD + \angle DAE = \angle BAE$$

(just like in solution 1), we obtain that the triangle $\triangle SBE$ is isosceles, so CO passes through S, because it is the perpendicular bisector of the segment BE.

Because $\angle BOC = \angle BAS$, we obtain that the quadrilateral $ASOB$ is cyclic, so $\angle BAL = \angle BSO$. Denote by L the intersection between AO and γ, then $\angle LDS = \angle BAL$. Combining these two equalities leads to $\angle BSO = \angle LDS$, so $LD \| SO$.

This means that LD is midline in triangle $\triangle AOC$, so L, which lies on γ, is the midpoint of the segment AO. □



Solution 4. Like in the previous solutions, establish that C is the midpoint of the segment BE.

Let F be the midpoint of the segment AB. Then both F and C lie on the circle of diameter BO. Because the quadrilateral $BFDC$ is cyclic, it means that D also lies on that circle.

Let K be the midpoint of BO. Then, AK must be the perpendicular bisector of the segment BC, so $AK \| OC$, which implies that $\angle KAD = \angle DCO$. However, $\angle DCO = \angle KBD$, because $DBCO$ is cyclic. From the two equalities we obtain that $\angle KAD = \angle KBD$, so K lies on γ. Furthermore, AK is the bisector of $\angle BAD$, so K is in fact the midpoint of the arc BD.

Now consider a reflection across OF. Clearly, B maps to A. Because OF is the perpendicular bisector of the segment AB, γ maps to itself through this reflection. Thus, K, the intersection between OB and γ, will map to L, the intersection between OA and γ. This implies that L is the midpoint of the segment AO. $\square$

2020 BMO, Problem 2

Denote $\mathbb{Z}_{>0} = \{1, 2, 3, \ldots\}$ the set of all positive integers. Determine all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that, for each positive integer n,

i) $\sum\limits_{k=1}^{n} f(k)$ is a perfect square, and

ii) $f(n)$ divides n^3.

Solution. Induct on n to show that $f(n) = n^3$ for all positive integers n. It is readily checked that this f satisfies the conditions in the statement. The base case, $n = 1$, is clear.

Let $n \geqslant 2$ and assume that $f(m) = m^3$ for all positive integers $m < n$. Then $\sum_{k=1}^{n-1} f(k) = \frac{n^2(n-1)^2}{4}$, and reference to the first condition in the statement yields

$$f(n) = \sum_{k=1}^{n} f(k) - \sum_{k=1}^{n-1} f(k) = \left(\frac{n(n-1)}{2} + k\right)^2 - \frac{n^2(n-1)^2}{4} = k(n^2 - n + k),$$

for some positive integer k.

The divisibility condition in the statement implies $k(n^2 - n + k) \leqslant n^3$, which is equivalent to $(n - k)(n^2 + k) \geqslant 0$, showing that $k \leqslant n$.

On the other hand, $n^2 - n + k$ must also divide n^3. But, if $k < n$, then

$$n < \frac{n^3}{n^2 - 1} \leqslant \frac{n^3}{n^2 - n + k} \leqslant \frac{n^3}{n^2 - n + 1} < \frac{n^3 + 1}{n^2 - n + 1} = n + 1,$$

therefore $\frac{n^3}{n^2-n+k}$ cannot be an integer.

Consequently, $k = n$, so $f(n) = n^3$. This completes induction and concludes the proof.

2020 BMO, Problem 2

Solution 2

Let $F(n) = f(1) + f(2) + \ldots + f(n)$. We use the following two observations:

Lemma 1 $F(n) \leq \left(\frac{n(n+1)}{2}\right)^2$

Proof: Since $f(i)|i^3$, for all i we have $f(i) \leq i^3$, and adding all up we get

$$f(1) + f(2) + \ldots + f(n) \leq 1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Lemma 2 $F(n) \geq n^2$

Proof: Note that $F(n)$ is injective and increasing since $f(i) > 0, \forall i$. Since $F(n)$ is a perfect square for all n the desired result is obtained.

Lemma 3 $f(p) = p^3$ for all p prime.

Proof: Since $f(p)|p^3$, the only possible values for $f(p)$ are $1, p, p^2, p^3$. We show that $f(p)$ can not be 1 or p or p^2.

Case 1: Suppose $f(p) = 1$. This implies $F(p-1)$ and $F(p)$ are two consecutive numbers, grater than 1 and perfect squares. This is impossible, contradiction.

Case 2: Suppose $f(p) = p$. Let $F(p-1) = a^2$ and $F(p) = b^2$. Hence we have $p = (a-b)(a+b)$, so a^2 has to be $\left(\frac{(p-1)}{2}\right)^2$ and b^2 has to be $\left(\frac{(p+1)}{2}\right)^2$. But by Lemma 2 we know $F(p-1) = a^2 \geq (p-1)^2$, contradiction.

Case 3: Suppose $f(p) = p^2$. Again we have $p^2 = (a-b)(a+b)$ and since $a, b > 0$ we have to have $a - b = 1$ and $a + b = p^2$. This gives $F(p-1) = \left(\frac{p^2-1}{2}\right)^2$. But from Lemma 2, we know $F(p-1) \leq \left(\frac{p^2-p}{2}\right)^2$, hence we get a contradiction again and $f(p)$ can not be p^2.

To finish the proof, we need to show that $f(n) = n^3$ for all nonprime values as well. Let $p > n$ be a prime. We know $f(p) = p^3$ and $f(p) = F(p) - F(p-1) = (a-b)(a+b)$. By a reasoning similar to Case 2 above, we can not have $a - b = 1$ and $a + b = p^3$ so we have to have $a - b = p$ and $a + b = p^2$. This gives us $F(p-1) = f(1) + f(2) + \ldots + f(p-1) = \left(\frac{p^2-p}{2}\right)$, so we have equality in Lemma 1. We know this only happens if $f(i) = i^3$ for all $i \leq p - 1$. This concludes the proof.

2020 BMO, Problem 3

Let k be a positive integer. Determine the least integer $n \geqslant k+1$ for which the game below can be played indefinitely:

> Consider n boxes, labelled b_1, b_2, …, b_n. For each index i, box b_i contains initially exactly i coins. At each step, the following three substeps are performed in order:
>
> **(1)** Choose $k+1$ boxes;
>
> **(2)** Of these $k+1$ boxes, choose k and remove at least half of the coins from each, and add to the remaining box, if labelled b_i, a number of i coins.
>
> **(3)** If one of the boxes is left empty, the game ends; otherwise, go to the next step.

Solution. The required minimum is $n = 2^k + k - 1$.

In this case the game can be played indefinitely by choosing the last $k+1$ boxes, b_{2^k-1}, b_{2^k}, …, b_{2^k+k-1}, at each step: At step r, if box b_{2^k+i-1} has exactly m_i coins, then $\lceil m_i/2 \rceil$ coins are removed from that box, unless $i \equiv r-1 \pmod{k+1}$, in which case $2^k + i - 1$ coins are added. Thus, after step r has been performed, box b_{2^k+i-1} contains exactly $\lfloor m_i/2 \rfloor$ coins, unless $i \equiv r-1 \pmod{k+1}$, in which case it contains exactly $m_i + 2^k + i - 1$ coins. This game goes on indefinitely, since each time a box is supplied, at least $2^k - 1$ coins are added, so it will then contain at least 2^k coins, good enough to survive the k steps to its next supply.

We now show that no smaller value of n works. So, let $n \leqslant 2^k + k - 2$ and suppose, if possible, that a game can be played indefinitely. Notice that a box currently containing exactly m coins survives at most $w = \lfloor \log_2 m \rfloor$ withdrawals; this w will be referred to as the *weight* of that box. The sum of the weigths of all boxes will referred to as the *total* weight. The argument hinges on the lemma below, proved at the end of the solution.

Lemma. *Performing a step does not increase the total weight. Moreover, supplying one of the first $2^k - 2$ boxes strictly decreases the total weight.*

Since the total weight cannot strictly decrease indefinitely, $n > 2^k - 2$, and from some stage on none of the first $2^k - 2$ boxes is ever supplied. Recall that each step involves a $(k+1)$-box choice. Since $n \leqslant 2^k + k - 2$, from that stage on, each step involves a withdrawal from at least one of the first $2^k - 2$ boxes. This cannot go on indefinitely, so the game must eventually come to an end, contradicting the assumption.

Consequently, a game that can be played indefinitely requires $n \geqslant 2^k + k - 1$.

Proof of the Lemma. Since a withdrawal from a box decreases its weight by at least 1, it is sufficient to show that supplying a box increases its weight by at most k; and if the latter is amongst the first $2^k - 2$ boxes, then its weight increases by at most $k-1$. Let the box to be supplied be b_i and let it currently contain exactly m_i coins, to proceed by case analysis:

If $m_i = 1$, the weight increases by $\lfloor \log_2(i+1) \rfloor \leqslant \lfloor \log_2(2^k + k - 1) \rfloor \leqslant \lfloor \log_2(2^{k+1} - 2) \rfloor \leqslant k$; and if, in addition, $i \leqslant 2^k - 2$, then the weight increases by $\lfloor \log_2(i+1) \rfloor \leqslant \lfloor \log_2(2^k - 1) \rfloor = k - 1$.

If $m_i = 2$, then the weight increases by $\lfloor \log_2(i+2) \rfloor - \lfloor \log_2 2 \rfloor \leqslant \lfloor \log_2(2^k + k) \rfloor - 1 \leqslant k - 1$.

If $m_i \geqslant 3$, then the weight increases by

$$\begin{aligned} \lfloor \log_2(i + m_i) \rfloor - \lfloor \log_2 m_i \rfloor &\leqslant \lfloor \log_2(i + m_i) - \log_2 m_i \rfloor + 1 \\ &\leqslant \left\lfloor \log_2 \left(1 + \frac{2^k + k - 2}{3} \right) \right\rfloor + 1 \leqslant k, \end{aligned}$$

since $1 + \frac{1}{3}(2^k + k - 2) = \frac{1}{3}(2^k + k + 1) < \frac{1}{3}(2^k + 2^{k+1}) = 2^k$.

Finally, let $i \leqslant 2^k - 2$ to consider the subcases $m_i = 3$ and $m_i \geqslant 4$. In the former subcase, the weight increases by

$$\lfloor \log_2(i+3) \rfloor - \lfloor \log_2 3 \rfloor \leqslant \lfloor \log_2(2^k + 1) \rfloor - 1 = k - 1,$$

and in the latter by

$$\begin{aligned} \lfloor \log_2(i + m_i) \rfloor - \lfloor \log_2 m_i \rfloor &\leqslant \lfloor \log_2(i + m_i) - \log_2 m_i \rfloor + 1 \\ &\leqslant \left\lfloor \log_2 \left(1 + \frac{2^k - 2}{4} \right) \right\rfloor + 1 \leqslant k - 1, \end{aligned}$$

since $1 + \frac{1}{4}(2^k - 2) = \frac{1}{4}(2^k + 2) < 2^{k-2} + 1$. This ends the proof and completes the solution.

2020 BMO, Problem 4

Let $a_1 = 2$ and, for every positive integer n, let a_{n+1} be the smallest integer strictly greater than a_n that has more positive divisors than a_n. Prove that $2a_{n+1} = 3a_n$ only for finitely many indices n.

Solution. Begin with a mere remark on the terms of the sequence under consideration.

Lemma 1. *Each a_n is minimal amongst all positive integers having the same number of positive divisors as a_n.*

Proof. Suppose, if possible, that for some n, some positive integer $b < a_n$ has as many positive divisors as a_n. Then $a_m < b \leqslant a_{m+1}$ for some $m < n$, and the definition of the sequence forces $b = a_{m+1}$. Since $b < a_n$, it follows that $m + 1 < n$, which is a contradiction, as a_{m+1} should have less positive divisors than a_n. □

Let $p_1 < p_2 < \cdots < p_n < \cdots$ be the strictly increasing sequence of prime numbers, and write canonical factorisations into primes in the form $N = \prod_{i\geqslant 1} p_i^{e_i}$, where $e_i \geqslant 0$ for all i, and $e_i = 0$ for all but finitely many indices i; in this notation, the number of positive divisors of N is $\tau(N) = \prod_{i\geqslant 1}(e_i + 1)$.

Lemma 2. *The exponents in the canonical factorisation of each a_n into primes form a non-strictly decreasing sequence.*

Proof. Indeed, if $e_i < e_j$ for some $i < j$ in the canonical decomposition of a_n into primes, then swapping the two exponents yields a smaller integer with the same number of positive divisors, contradicting Lemma 1. □

We are now in a position to prove the required result. For convenience, a term a_n satisfying $3a_n = 2a_{n+1}$ will be referred to as a *special* term of the sequence.

Suppose now, if possible, that the sequence has infinitely many special terms, so the latter form a strictly increasing, and hence unbounded, subsequence. To reach a contradiction, it is sufficient to show that:

- **(1)** The exponents of the primes in the factorisation of special terms have a common upper bound e; and
- **(2)** For all large enough primes p, no special term is divisible by p.

Refer to Lemma 2 to write $a_n = \prod_{i\geqslant 1} p_i^{e_i(n)}$, where $e_i(n) \geqslant e_{i+1}(n)$ for all i.

Statement **(2)** is a straightforward consequence of **(1)** and Lemma 1. Suppose, if possible, that some special term a_n is divisible by a prime $p_i > 2^{e+1}$, where e is the integer provided by **(1)**. Then $e \geqslant e_i(n) > 0$, so $2^{e_1(n)e_i(n)+e_i(n)}a_n/p_i^{e_i(n)}$ is a positive integer with the same number of positive divisors as a_n, but smaller than a_n. This contradicts Lemma 1. Consequently, no special term is divisible by a prime exceeding 2^{e+1}.

To prove **(1)**, it is sufficient to show that, as a_n runs through the special terms, the exponents $e_1(n)$ are bounded from above. Then, Lemma 2 shows that such an upper bound e suits all primes.

Consider a large enough special a_n. The condition $\tau(a_n) < \tau(a_{n+1})$ is then equivalent to $(e_1(n)+1)(e_2(n)+1) < e_1(n)(e_2(n)+2)$. Alternatively, but equivalently, $e_1(n) \geqslant e_2(n)+2$. The latter implies that a_n is divisible by 8, for either $e_1(n) \geqslant 3$ or a_n is a large enough power of 2.

Next, note that $9a_n/8$ is an integer strictly between a_n and a_{n+1}, so $\tau(9a_n/8) \leqslant \tau(a_n)$, which is equivalent to

$$(e_1(n)-2)(e_2(n)+3) \leqslant (e_1(n)+1)(e_2(n)+1),$$

so $2e_1(n) \leqslant 3e_2(n)+7$. This shows that a_n is divisible by 3, for otherwise, letting a_n run through the special terms, 3 would be an upper bound for all but finitely many $e_1(n)$, and the special terms would therefore form a bounded sequence.

Thus, $4a_n/3$ is another integer strictly between a_n and a_{n+1}. As before, $\tau(4a_n/3) \leqslant \tau(a_n)$. Alternatively, but equivalently,

$$(e_1(n)+3)e_2(n) \leqslant (e_1(n)+1)(e_2(n)+1),$$

so $2e_2(n)-1 \leqslant e_1(n)$. Combine this with the inequality in the previous paragraph to write $4e_2(n)-2 \leqslant 2e_1(n) \leqslant 3e_2(n)+7$ and infer that $e_2(n) \leqslant 9$. Consequently, $2e_1(n) \leqslant 3e_2(n)+7 \leqslant 34$, showing that $e = 17$ is suitable for **(1)** to hold. This establishes **(1)** and completes the solution.